

ASYMPTOTIC BEHAVIOUR AND PROPAGATION PROPERTIES OF THE ONE-DIMENSIONAL FLOW OF GAS IN A POROUS MEDIUM

BY

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ABSTRACT. The one-dimensional porous media equation $u_t = (u^m)_{xx}$, $m > 1$, is considered for $x \in \mathbf{R}$, $t > 0$ with initial conditions $u(x, 0) = u_0(x)$ integrable, nonnegative and with compact support. We study the behaviour of the solutions as $t \rightarrow \infty$ proving that the expressions for the density, pressure, local velocity and interfaces converge to those of a model solution. In particular the first term in the asymptotic development of the free-boundary is obtained.

0. Introduction. Suppose we have a certain distribution of gas whose density at time $t = 0$ is given by a function $u_0(x)$ of one spatial direction ($x \in \mathbf{R}$). If the gas flows through a homogeneous porous medium the *density* $u = u(x, t)$ at time $t > 0$ is governed by the equation

$$(0.1) \quad u_t = (u^m)_{xx}$$

for $x \in \mathbf{R}$ and $t > 0$; m is a physical constant, $m > 1$, and we have scaled out other physical constants (see [1] for a physical derivation). u satisfies the initial condition

$$(0.2) \quad u(x, 0) = u_0(x)$$

where u_0 satisfies the following assumptions:

$$(0.3) \quad u_0 \in L^1(\mathbf{R}), \quad u_0 \geq 0, u_0 \not\equiv 0,$$

and u_0 is compactly supported, i.e. if $\Omega_0 = \{x \in \mathbf{R}: u_0(x) > 0\}$ we have

$$(0.4) \quad a_1 = \text{ess inf } \Omega_0 > -\infty, \quad a_2 = \text{ess sup } \Omega_0 < \infty.$$

Sticking to the above application we define the pressure by $v = mu^{m-1}/(m-1)$ on $Q = \mathbf{R} \times (0, \infty)$ and the local velocity by $V = -v_x$ on the *domain of dependence*

$$(0.5) \quad \Omega = \Omega[u] = \{(x, t) \in Q: u(x, t) > 0\}.$$

The *total mass* at time $t > 0$ is $M(t) = \int u(x, t) dx$ and the *center of mass* is $x_c(t) = M(t)^{-1} \int u(x, t)x dx$. Set $M_0 = \int u_0(x) dx$ and $x_0 = M_0^{-1} \int u_0(x)x dx$: $M_0 > 0$ and $a_1 < x_0 < a_2$. $l_0 = a_2 - a_1$ measures the *dispersion* of the initial data.

Much is already known for problem (0.1)–(0.4); see [19] for a survey of results up to 1980, where the n -dimensional case is considered, $n \geq 1$. In particular (0.1)–(0.4)

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admits a unique continuous weak solution $u(x, t) \geq 0$ [18], [3], such that for $t > 0$, $u(\cdot, t)$ has compact support [14]. Thus $\Omega(t) = \{x \in \mathbf{R}: u(x, t) > 0\}$ is bounded for every $t > 0$ and two *outer interfaces* arise with equations $x = \xi_i(t)$, $i = 1, 2$, $t \geq 0$, where

$$(0.6) \quad \begin{aligned} \xi_1(t) &= \inf \Omega(t) & \text{if } t > 0, & \quad \xi_1(0) = a_1, \\ \xi_2(t) &= \sup \Omega(t) & \text{if } t > 0, & \quad \xi_2(0) = a_2. \end{aligned}$$

As a consequence of the inequality [10]

$$(0.7) \quad u_t \geq -u/(m+1)t,$$

the set $\{\Omega(t): t > 0\}$ is ordered by inclusion: $\Omega(t') \supset \Omega(t)$ if $t' > t$ so that $(-1)^i \xi_i$ is a nondecreasing function. Moreover there exist $t_i^* \geq 0$ (called *waiting-times*) such that for $0 \leq t \leq t_i^*$, $\xi_i(t) = a_i$ [16] and, for $t > t_i^*$, $\xi_i(t)$ is continuously differentiable and $(-1)^i \xi_i'(t) > 0$ (once the interface starts to move it never stops) [11]. [16] proves that when $t \rightarrow \infty$ $(-1)^i \xi_i(t)$ behaves like $t^{1/(m+1)}$. These results are proved under the simplifying hypotheses that u_0 is continuous, $u_0(x) > 0$ for every $x \in I = (a_1, a_2)$ and vanishes outside I , but the proofs apply under conditions (0.3), (0.4). On the contrary under only these two conditions the property that $\Omega(t) = (\xi_1(t), \xi_2(t))$ does not hold in general, i.e. *inner interfaces* can appear that make $\Omega(t)$ disconnected for some time interval $0 \leq t \leq T$, $T > 0$.

We shall be concerned in this paper with the following question: Give significant information about the behaviour of the solutions to (0.1)–(0.4) in terms of a simple information on u_0 , specifically in terms of M_0, x_0, a_1, a_2 . Our contribution deals with the asymptotic behaviour of density, pressure, velocity and free-boundaries and on the global properties of Ω .

To describe the large-time behaviour we take as *model solutions* the class of explicit self-similar solutions corresponding to an initial “instantaneous source” given by Barenblatt in 1952 [6], i.e. solutions of (0.1) with initial data

$$u_0(x) = M\delta(x - a)$$

where $M > 0$, $a \in \mathbf{R}$ and δ is Dirac’s delta function. The unique weak solution $\bar{u}(x, t; M, a)$ is given in terms of its pressure by

$$(0.8) \quad \bar{v}(x, t; M, a) = [r(t)^2 - (x - a)^2]_+/2(m+1)t$$

where

$$(0.9) \quad r(t) = c_m(M^{m-1}t)^{1/(m+1)}$$

$$\text{with } c_m = \left\{ \frac{2m(m+1)}{m-1} B\left(\frac{m}{m-1}, \frac{1}{2}\right)^{1-m} \right\}^{1/(m+1)},$$

$B(\cdot, \cdot)$ being Euler’s Beta function ($[s]_+$ means $\max(s, 0)$). Its interfaces are the strictly monotone C^∞ -curves given by $\xi_i(t) = (-1)^i r(t) + a$, $i = 1, 2$, and the velocity is defined on $\Omega[\bar{u}] = \{(x, t) \in Q: |x - a| < r(t)\}$ by $\bar{V}(x, t) = (x - a) \cdot ((m+1)t)^{-1}$.

Let $\bar{u}(x, t; M) = \bar{u}(x, t; M, 0)$; then $\bar{u}(x, t; M, a) = \bar{u}(x - a, t; M)$.

Our main result shows to what extent the solution $u(x, t)$ to (0.1)–(0.4) resembles the self-similar $\bar{u} = \bar{u}(x, t; M_0, x_0)$ with same mass and center of mass.

THEOREM A. *Let $u(x, t)$ be the solution to (0.1)–(0.4) and let $\bar{u} = \bar{u}(x, t; M_0, x_0)$. Then:*

(i) *for every $t \geq T^* = (l_0/c_m)^{m+1} M_0^{1-m}$, $\Omega(t)$ is the open interval $(\xi_1(t), \xi_2(t))$ with strictly expanding borders;*

(ii) *as $t \rightarrow \infty$ we have*

$$(0.10) \quad (-1)^i (\xi_i(t) - \bar{\xi}_i(t)) \downarrow 0;$$

$$(0.11) \quad \xi'_i(t)/\bar{\xi}'_i(t) \uparrow 1 \quad \text{and} \quad t |\xi'_i(t) - \bar{\xi}'_i(t)| \rightarrow 0;$$

$$(0.12) \quad t |V(x, t) - (x - x_0)/(m+1)t| \rightarrow 0 \quad \text{uniformly in } x \in \Omega(t),$$

$$(0.13) \quad t^{m/(m+1)} |v(x, t) - \bar{v}(x, t)| \rightarrow 0 \quad \text{uniformly in } x \in \mathbf{R}$$

and for every $x \in \mathbf{R}$, $t > 0$

$$(0.13)' \quad v(x, t) \leq \max_{x \in \mathbf{R}} \bar{v}(x, t) = c_m^2 (2(m+1))^{-1} (M_0^2 t^{-1})^{(m-1)/(m+1)}.$$

We may write (0.10), (0.11) as giving the first term in the asymptotic development of $\xi_i(t)$ and $\xi'_i(t)$:

$$(0.14) \quad \xi_i(t) = x_0 + (-1)^i c_m M_0^{(m-1)/(m+1)} t^{1/(m+1)} + o(1),$$

$$(0.15) \quad \xi'_i(t) = (-1)^i (c_m/(m+1)) M_0^{(m-1)/(m+1)} t^{m/(m+1)} + o(1/t)$$

where $o(1)$ and $o(1/t)$ are the usual Landau o 's taken as $t \rightarrow \infty$. Theorem A shows that M_0 and x_0 are the only relevant initial data in the first approximation to the large-time behaviour of the solutions to (0.1)–(0.4). In particular (0.13) implies for $1 < m \leq 2$ the estimate

$$(0.16) \quad t^{2/(m+1)} |u(x, t) - \bar{u}(x, t)| \rightarrow 0$$

uniformly in $x \in \mathbf{R}$. If $m > 2$, however, (0.16) holds uniformly in $x: |x - x_0| \leq \alpha r(t; M_0)$ for every $0 < \alpha < 1$ and we obtain $t^\sigma |u(x, t) - \bar{u}(x, t)| \rightarrow 0$ at $t \rightarrow \infty$ uniformly in $x \in \mathbf{R}$ for $\sigma = m/(m^2 - 1)$.

As a precedent to these results Kamin [15] proved the convergence of u towards a self-similar \bar{u} with equal mass with an estimate

$$(0.17) \quad t^{1/(m+1)} |u(x, t) - \bar{u}(x, t)| \rightarrow 0$$

that does not allow for the characterization of x_0 . Friedman and Kamin [13] extend (0.17) to dimensions $n \geq 1$. Several terms of the asymptotic representation of u were stated in [7] without proof.

We begin by reviewing in §1 several properties of the solutions. In particular we prove the time-invariance of the mass and the center of mass, i.e. for every $t > 0$, $M(t) = M_0$, $x_c(t) = x_0$ (Lemma 1.1).

In §2 we introduce a comparison principle, based on the evaluation of masses, that we name "Shifting-Comparison Principle" (Sh.C.P.) (Lemma 3.2). As immediate corollaries we derive the estimate for the free boundaries $(-1)^i \xi'_i(t) t^{-1/(m+1)} \rightarrow c_m M_0^{(m-1)/(m+1)}$, which improves Knerr's result [16], and the estimate in Theorem

A(i), where we remark that T^* is optimal in terms of M_0 and l_0 as an upper bound for both the occurrence of waiting-times and that of an inner free-boundary.

§3 is devoted to proving Theorem A. As a main ingredient we use a sharp version of Caffarelli and Friedman's [11] differential inequality for the interfaces that in fact gives the monotonicity of $\zeta'_i(t)t^{m/(m+1)}$ (Lemma 3.1).

The case where u_0 is a symmetric function is considered in §4. Then $u(x, t)$ is symmetric with respect to x and we prove *optimal* rates of convergence in the results of Theorem A by means of a new "Concentration-Comparison Principle" (Theorem B).

Finally §5 considers the right interface $\zeta(t)$ of a solution of (0.1), (0.2) with u_0 satisfying (0.3) and, instead of (0.4),

$$(0.18) \quad \text{ess sup } \Omega_0 = 0.$$

The behaviour of ζ as $t \rightarrow \infty$ and $t \rightarrow 0$ is investigated as well as its dependence on the L^p -norm of the initial data u_0 , $1 \leq p \leq \infty$.

Let us remark that the asymptotic behaviour of the porous medium equation in bounded domains of \mathbf{R}^n has been studied recently by Aronson and Peletier [5].

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1. Preliminaries.

1.1 *Existence of solutions.* We begin by reviewing the existence and properties of weak solutions to (0.1), (0.2). It is known [3] that for every $u_0 \in L^1(\mathbf{R})$, $u_0 \geq 0$, there exists a unique continuous function $u = u(x, t)$ in $Q = \mathbf{R} \times (0, \infty)$ with the following properties:

- $$(1.1) \quad \begin{aligned} & \text{(i)} \quad u \in C([0, \infty); L^1(\mathbf{R})) \cap L^\infty(\mathbf{R} \times [\delta, \infty)) \text{ for every } \delta > 0, \\ & \text{(ii)} \quad u_t = (u^m)_{xx} \text{ in the sense of distributions on } Q, \\ & \text{(iii)} \quad u(0) = u_0 \text{ in } L^1(\mathbf{R}). \end{aligned}$$

Here $u(t)$ denotes the element $u(\cdot, t)$ in $L^1(\mathbf{R})$. Moreover u_t and $(u^m)_{xx}$ exist a.e., $u_t \in L^1_{\text{loc}}(Q)$ satisfies (0.7) and

$$(1.2) \quad \|u_t(\cdot, t)\|_1 \leq (2/(m+1)t)\|u_0\|_1.$$

To obtain the solution we may approximate u_0 by a decreasing sequence of strictly positive, smooth functions u_0^n , apply to u_0^n the existence theorem of [18] and derive (1.1) in the limit.

In §2 we shall need an alternative approach: we discretize (0.1) in time and have recourse to Crandall and Liggett's Generation Theorem (see [12]). In fact given a continuous increasing function $\phi: \mathbf{R} \rightarrow \mathbf{R}$, $\phi(0) = 0$, the operator $A = A_\phi$ defined on

$$(1.3) \quad D(A) = \{u \in L^1(\mathbf{R}): \phi(u) \in W^{1,\infty}(\mathbf{R}) \text{ and } \phi(u)_{xx} \in L^1(\mathbf{R})\}$$

by

$$(1.4) \quad Au = -\phi(u)_{xx} \quad \text{if } u \in D(A)$$

is m -accretive in $L^1(\mathbf{R})$, i.e. the resolvent $(I + \lambda A)^{-1}$ is a contraction on $L^1(\mathbf{R})$ for any $\lambda > 0$ [9], and the closure of $D(A)$ in $L^1(\mathbf{R})$ is $L^1(\mathbf{R})$ [10]. Hence the formula

$$(1.5) \quad S(t)u_0 = \lim_{n \rightarrow \infty} (I + (t/n)A)^{-n}u_0 \quad \text{for } t \geq 0, u_0 \in L^1(\mathbf{R}),$$

defines a *semigroup of contractions* $S(t) = S_\phi(t)$, $t \geq 0$, in $L^1(\mathbf{R})$. B nilan has proved [8] that $u(x, t) = (S(t)u_0)(x)$ solves in a generalized sense (called integral or mild sense) the evolution problem

$$(P_\phi) \quad \begin{cases} u_t = \phi(u)_{xx}, & x \in \mathbf{R}, t > 0, \\ u(x, 0) = u_0(x) \in L^1(\mathbf{R}), \end{cases}$$

and that these mild solutions are unique.

Setting $\phi(t) = s |s|^{m-1}$ we recover problem (0.1), (0.2) and both constructions give the same unique solution satisfying (1.1).

Since for every $f_i \in L^1(\mathbf{R})$, $i = 1, 2$, and $\lambda > 0$ we have [9]

$$(1.6) \quad \left\| \left[(I + \lambda A)^{-1}f_1 - (I + \lambda A)^{-1}f_2 \right]_+ \right\|_1 \leq \| [f_1 - f_2]_+ \|_1,$$

using (1.5) we obtain for every $u_i \in L^1(\mathbf{R})$, $i = 1, 2$,

$$(1.7) \quad \left\| [S(t)u_1 - S(t)u_2]_+ \right\|_1 \leq \| [u_1 - u_2]_+ \|_1.$$

(1.7) implies obvious comparison results for the solutions of (0.1), (0.2).

REMARK. Since the solutions $\bar{u}(x, t; M, a)$ are limits of solutions of (0.1) with initial data $u_0^n \geq 0$, $u_0^n \in L^1(\mathbf{R})$, such that $u_0^n \rightarrow M\delta(x)$ in the w^* -topology of $\mathfrak{M}(\mathbf{R})$, the space of bounded Radon measures (take $u_0^n(x) = \bar{u}(x, 1/n; M, a)$), the comparison results valid for solutions with L^1 -data apply also to \bar{u} .

1.2. *Group of transformations.* Equation (0.1) admits the biparametric group of transformations

$$(1.8) \quad \hat{u} = ku, \quad \hat{x} = L^{-1}x, \quad \hat{t} = k^{1-m}L^{-2}t,$$

i.e. if $u(x, t)$ is a solution of (0.1) with initial condition $u_0(x)$ then for every $k, L > 0$, $\hat{u}(x, t)$ defined by

$$(1.9) \quad \hat{u}(x, t) = ku(Lx, L^2k^{m-1}t)$$

is a solution with initial condition $\hat{u}_0(x) = ku_0(Lx)$. We write $\hat{u} = T_{K,L}u$. The transformation $T_{K,L}$ preserves the (initial) mass iff $k = L$.

We can use the group of transformations $T_{K,L}$ to reduce a problem (0.1), (1.2) to a simple one. Thus if $u(x, t)$ is such that $\int u_0(x) dx = M$ and we put $u = T_{M,1}\hat{u}$ we have $\hat{M} = \int \hat{u}_0(x) dx = 1$. By means of $T_{M,1}$ our conclusions on \hat{u} apply to u . For instance the respective free boundaries $\zeta(t)$ and $\hat{\zeta}(t)$ are related by

$$(1.10) \quad \zeta(t) = \hat{\zeta}(M^{m-1}t).$$

Notice that the self-similar solutions $\bar{u}(x, t; M)$ centered at $a = 0$ are invariant under $T_{k,k}$, $k > 0$, i.e.

$$(1.11) \quad \bar{u}(x, t; M) = k\bar{u}(kx, k^{m+1}t; M).$$

1.3. *Two invariants.* We establish here the invariance of the total mass M and the center of mass x_c for the solutions of (0.1)–(0.4):

LEMMA 1.1. *For every $t > 0$ $M(t) = M_0$ and $x_c(t) = x_0$.*

PROOF. Assume first that u_0 is also continuous, positive on $I = (a_1, a_2)$ and such that $u_0(x) > |x - a_i|^{1/(m-1)}$ in a neighbourhood of both a_1 and a_2 (so that the waiting-times t_1^* and t_2^* vanish, see [16]). Then $u \in C^\infty(\Omega)$ and $v \in C^1(\bar{\Omega} \cap Q)$, see [11]. It follows that $u^m \in C^1(\bar{\Omega} \cap Q)$ and $(u^m)_x$ vanishes on both interfaces.

Take now two arbitrary times $t_2 > t_1 > 0$ and set $G = \{(x, t): t_1 \leq t \leq t_2 \text{ and } \xi_1(t) \leq x \leq \xi_2(t)\}$. Then

$$\begin{aligned} \int_{\mathbf{R}} u(x, t_2) x \, dx - \int_{\mathbf{R}} u(x, t_1) x \, dx &= \int_G u_t \, dx \, dt \\ &= \int_{t_1}^{t_2} dt [(u^m)_x(\xi_2(t), t) - (u^m)_x(\xi_1(t), t)] = 0, \end{aligned}$$

$$\int_{\mathbf{R}} u(x, t_2) x \, dx - \int_{\mathbf{R}} u(x, t_1) x \, dx = \int_G x u_t \, dx = \int_{t_1}^{t_2} dt [x(u^m)_x - u^m]_{\{\xi_i(t), t\}}^{\{\xi_2(t), t\}} = 0,$$

and the result follows. For general u_0 approximate by a decreasing sequence $\{u^n\}$ as above and pass to the limit using the L^1 -continuity of the map $u_0 \mapsto u(t)$ (formula (1.7)). #

REMARKS. (1) The result is valid in a much more general context: for instance for the solutions of (P_ϕ) , without the restriction of nonnegativity.

(2) The invariance of the total mass has been widely used in connection with this problem: [10, 15], The invariance of the center of mass has been pointed out in [7].

1.4. *Regularity up to the interfaces.* We know that the solutions are classical in Ω . [1] proves that $v(x, t)$ is Lipschitz-continuous in x in $Q_\tau = \mathbf{R} \times (\tau, \infty)$ for every $\tau > 0$. But v_x need not be continuous at the interfaces (check the self-similar solutions). However [16] proves that $v_x(\xi_i(t), t)$ exists for every $t > 0$ as the limit of $v_x(x, t)$, $x \rightarrow \xi_i(t)$, $x \in \Omega(t)$ and

$$(1.12) \quad V(\xi_i(t), t) \equiv -v_x(\xi_i(t), t) = \xi_i'(t)$$

where $\xi_i'(t)$ is the right derivative of ξ at t . [11] proves that v_x is continuously differentiable up to the boundary $x = \xi_i(t)$ if $t > t_i^*$.

2. Comparison by shifting.

2.1. We introduce in this section a “Shifting-Comparison Principle” that allows us to compare a solution with given initial condition with the one corresponding to a displaced initial condition. To measure the relative displacement we use the corresponding *distribution functions* defined by

$$(2.1) \quad U(x, t) = \int_{-\infty}^x u(x, t) \, dx = \text{amount of mass in } (-\infty, x].$$

The idea behind the principle is that it is more feasible to compare masses than to compare point densities. The principle is in fact a maximum principle for the “integrated” equation $U_t = (U_x | U_x |^{m-1})_x$.

We prove the principle in an elliptic version. Then (1.5) allows us to derive the evolution version.

LEMMA 2.1 (SHIFTING COMPARISON PRINCIPLE. ELLIPTIC VERSION). *Let β be a continuous nondecreasing function such that $0 = \beta(0) \subset \text{Int } \beta(\mathbf{R})$ and let $f_i, i = 1, 2$, be integrable functions such that for every $x \in \mathbf{R}$,*

$$(2.2) \quad \int_{-\infty}^x f_1(x) dx \leq \int_{-\infty}^x f_2(x) dx.$$

Let u_i be the solution (E): $-u'' + \beta(u) = f_i$, with $u_i \in W^{1,\infty}(\mathbf{R})$ and $w_i = \beta(u_i) \in L^1(\mathbf{R})$ (see [9]). Then for every $x \in \mathbf{R}$,

$$(2.3) \quad \int_{-\infty}^x w_1(x) dx \leq \int_{-\infty}^x w_2(x) dx.$$

PROOF. Set $F_i(x) = \int_{-\infty}^x f_i(x) dx$ and $W_i(x) = \int_{-\infty}^x w_i(x) dx$. Assume that (2.3) does not hold so that $G = \{x \in \mathbf{R}: W_1(x) > W_2(x)\}$ is nonvoid. Let $I = (a, b)$, $-\infty \leq a < b \leq \infty$, be a maximal interval in G . For every $x \in I$ we have by integration of (E):

$$(2.4) \quad u'_1(x) = W_1(x) - F_1(x) > W_2(x) - F_2(x) = u'_2(x)$$

so that $u_1 - u_2$ is strictly increasing on I .

Assume now that $a > -\infty$. Then by continuity $W_1(a) = W_2(a)$ and $u_1(a) > u_2(a)$ (if $u_1(a) < u_2(a)$ we would have $u_1(x) < u_2(x)$ if $|x - a| < \varepsilon$ for an $\varepsilon > 0$ so that $w_1(x) \leq w_2(x)$, hence $W_1 - W_2$ is nonincreasing in $a - \varepsilon < x < a + \varepsilon$, contradicting the definition of a). By (2.4) we have $u_1 > u_2$ on I , so that $W_1 - W_2$ is nondecreasing on I . This implies that $b = \infty$ and $W_1(\infty) > W_2(\infty)$. But this contradicts the fact that $F_1(\infty) \leq F_2(\infty)$ and that $W_i(\infty) = F_i(\infty)$ for the solutions of (E) (see [9, formula (4.3)]).

If $a = -\infty$, $W_i(-\infty) = 0$ and $u_1(-\infty) \geq u_2(-\infty)$ by the preceding argument. Hence the same conclusion holds. #

REMARK. It is clear that the proof of Lemma 2.1 applies to much more general situations. In particular it is true for the solutions of

$$(2.5) \quad -\frac{d}{dx}A(x, u'(x)) + B(x, u(x)) = f(x)$$

where A and B are, say, increasing in u and continuous in x , and $f \in L^1(\mathbf{R})$.

Besides it holds for suitable Dirichlet or Neumann boundary conditions if (2.5) is posed in a bounded interval or a half-line.

Similar remarks apply to Lemma 2.2 to follow. #

The change of variables $w = \beta(u)$, $u = \phi(w)$ transforms $-u'' + \beta(u) = f$ into $-\phi(w)'' + w = f$, that can be written as $w = (I + A_\phi)^{-1}(f)$. Therefore setting $\phi(s) = s|s|^{m-1}$, i.e. $\beta(s) = s|s|^{(1-m)/m}$, we derive via (1.5) the following evolution version for the solutions of (0.1), (0.2):

LEMMA 2.2 (SHIFTING-COMPARISON PRINCIPLE. PARABOLIC VERSION). *Let $u^1(x, t)$, $u^2(x, t)$ be solutions of (0.1), (0.2) with initial data $u_0^1(x)$, $u_0^2(x) \in L^1(\mathbf{R})$. If for every*

$x \in \mathbf{R}$ we have

$$(2.6) \quad \int_{-\infty}^x u_0^1(x) dx \leq \int_{-\infty}^x u_0^2(x) dx, \quad \text{i.e.} \quad U_0^1(x) \leq U_0^2(x),$$

then for every $t > 0$ and every $x \in \mathbf{R}$,

$$(2.7) \quad \int_{-\infty}^x u^1(x, t) dx \leq \int_{-\infty}^x u^2(x, t) dx, \quad \text{i.e.} \quad U^1(x, t) \leq U^2(x, t).$$

REMARKS. (1) We say that the mass of u^1 is *shifted to the right* with respect to that of u^2 at time $t = 0$; this situation is preserved for every $t > 0$.

(2) As said in §1.1 we may consider initial data of the form $M\delta(x - x_0)$.

2.2. *First applications.* An easy application of the Sh.C.P., comparing with self-similar solutions that concentrate all the mass M_0 of u_0 at the extreme points $x = a_1$ or $x = a_2$, allows us to bound the right and left interface of u from above and below, giving a first estimate of their asymptotic behaviour. Here and in the sequel we fix $M = M_0$ in $r(t)$: $r(t) = c_m(M_0^{m-1}t)^{1/(m+1)}$.

COROLLARY 2.3. *For every $t > 0$ we have*

$$(2.8) \quad a_1 - r(t) < \zeta_1(t) < a_2 - r(t), \quad a_1 + r(t) < \zeta_2(t) < a_2 + r(t),$$

so that

$$(2.9) \quad \lim_{t \rightarrow \infty} \zeta_i(t)t^{-1/(m+1)} = (-1)^i c_m M_0^{(m-1)/(m+1)}.$$

PROOF. We take $u^1(x, t) = \bar{u}(x, t; M_0, a_1)$ and $u^2(x, t) = \bar{u}(x, t; M_0, a_2)$. Since we have $U_0^1(x) \geq U_0(x) \geq U_0^2(x)$, we conclude that $U^1(x, t) \geq U(x, t) \geq U^2(x, t)$. But since the interfaces can be characterized in terms of U : $\zeta_1(t) = \inf\{x \in \mathbf{R}: U(x, t) > 0\}$, $\zeta_2(t) = \sup\{x \in \mathbf{R}: U(x, t) < M_0\}$, it follows that $\zeta_1'(t) = a_1 - r(t) \leq \zeta_1(t) \leq \zeta_1^2(t) = a_2 - r(t)$ and likewise for $\zeta_2(t)$.

We must prove that the inequalities (2.8) are strict: assume for instance that for a $t_0 > 0$, $\zeta_2(t_0) = a_2 + r(t_0)$. Since $\zeta_2(t) \leq a_2 + r(t)$ for every $t > 0$, we have $\zeta_2'(t_0) = r'(t_0)$. Take now $\tilde{u}(x, t) = \bar{u}(x - a_2, t; M_0)$. For $t = t_0$ we have $\zeta_2(t_0) = \tilde{\zeta}_2(t_0)$, $\zeta_2'(t) = \tilde{\zeta}_2'(t)$. Using the fact that $v_x(\zeta(t), t) = -\zeta'(t)$ for any $t > 0$ [16], and that $v_{xx} \geq 0 - 1/(m+1)t = \tilde{v}_{xx}$ (in $\Omega[\tilde{u}]$) [3], we conclude that, at $t = t_0$, $u(x, t_0) \geq \tilde{u}(x, t_0)$. Since both have mass M_0 it follows that $u = \tilde{u}$ at $t = t_0$. But this is impossible since $x_0 < a_2 = \tilde{x}_0$ and the center of mass is invariant. #

As a consequence of (2.8) we bound above the waiting-times t_i^* in terms of $M_0 = \|u_0\|_1$ and $l_0 = a_2 - a_1$:

COROLLARY 2.4. *We have*

$$(2.10) \quad t_i^* < T^* \equiv (l_0/c_m)^{m+1} M_0^{1-m}.$$

PROOF. For $t \geq T^*$, $\zeta_2(t) > a_1 + r(T^*) = a_1 + l_0 = a_2$. Similarly for ζ_1 . #

REMARKS. (1) (2.10) is sharp in terms of M_0 and l_0 . To see this choose an initial datum u_0 with two components: one, $u_0^1(x)$, of mass $\geq M_0 - \varepsilon$, $\varepsilon > 0$ small, supported in $[a_1, a_1 + \varepsilon]$, and the other, $u_0^2(x)$, of mass obviously $\leq \varepsilon$, supported in $[a_1 - \varepsilon, a_2]$, and such that the corresponding solution $u^2(x, t)$ has vertical interfaces for at least a time T^* . Up to the time where the interfaces of the solutions to both

partial initial data, $u^1(x, t)$ and $u^2(x, t)$, meet we have $u(x, t) = u^1(x, t) + u^2(x, t)$ so that $\zeta(t) = a_2$. But as $\varepsilon \rightarrow 0$ this time is easily seen to approach T^* (use the Sh.C.P.).

(2) For recent work on the determination of the waiting-times see [4 and 17]. #

We now turn to the existence of an inner free-boundary, Γ_{in} . Since we are mainly interested in the large-time behaviour we want to bound above the time at which the inner free-boundary ceases to exist. For that we define on \mathbf{R} the nonnegative function

$$(2.11) \quad \tau(x) = \sup\{t > 0: u(x, t) = 0\}.$$

It is clear that $\Omega[u] = \{(x, t) \in \mathbf{Q}: t > \tau(x)\}$. We have as a consequence of Corollary 2.4:

COROLLARY 2.5. *For every $x \in [a_1, a_2]$, $\tau(x) < T^*$ so that*

$$(2.1.2) \quad \Gamma_{\text{in}} \subset (a_1, a_2) \times (0, T^*).$$

PROOF. If $x = a_1$ or $x = a_2$, $\tau(a_i) = t_i^*$ and we are reduced to (2.10). For any \bar{x} : $a_1 < \bar{x} < a_2$ such that $\tau(\bar{x}) > 0$, we write $u_0(x)$ as $u_0 = u_0^{(1)} + u_0^{(2)}$ with $u_0^{(1)} = u_0 \cdot \chi_{(-\infty, \bar{x}]}$ and $u_0^{(2)} = u_0 \cdot \chi_{(\bar{x}, \infty)}$, χ_E denoting the characteristic function of a set $E \subset \mathbf{R}$. Let $u^{(1)}, u^{(2)}$ be the respective solutions. By comparison $u(x, t) \geq u^{(i)}(x, t)$ so that $\tau(\bar{x}) \leq \tau^{(i)}(\bar{x})$, $i = 1, 2$.

Now observe that $\tau^{(1)}(\bar{x}) = t_2^{*(1)}$ is the right waiting-time of $u^{(1)}$ and $\tau^{(2)}(\bar{x}) = t_1^{*(2)}$ is the left waiting-time of $u^{(2)}$, so that

$$(2.13) \quad \tau(\bar{x}) < \left(\frac{\bar{x} - a_1}{c_m} \right)^{m+1} M_1^{1-m} \quad \text{with } M_1 = \int_{a_1}^{\bar{x}} u_0(x) dx$$

and

$$(2.14) \quad \tau(\bar{x}) < \left(\frac{a_2 - \bar{x}}{c_m} \right)^{m+1} M_2^{1-m} \quad \text{with } M_2 = M_0 - M_1.$$

It follows from (2.13), (2.14) that

$$\begin{aligned} l_0 &= (a_2 - \bar{x}) + (\bar{x} - a_1) > c_m (M_1^{(m-1)/(m+1)} + M_2^{(m-1)/(m+1)}) \tau(\bar{x}) \\ &\geq c_m M_0^{(m-1)/(m+1)} \tau(\bar{x}) \end{aligned}$$

and the result follows. #

REMARKS. (1) (2.12) is sharp: argue as in Remark (1) to Corollary 2.2.

(2) It is not difficult to see that

$$(2.15)$$

$$\Gamma_{\text{in}} = \{(x, \tau(x)): x \in \text{Int}(\Lambda)\} \cup \{(x, t): x \in \partial\Lambda \cap \Lambda \text{ and } 0 < t \leq \tau(x)\}$$

where $\Lambda = \{(a_1, a_2): \tau(x) > 0\}$ and that for every maximal open interval I in Λ , $\Gamma_{\text{in}} \cap (I \times (0, \infty))$ consists of one or two monotone C^1 -arcs (pieces of interfaces to subsolutions as above). #

3. The asymptotic behaviour. This section is devoted to proving Theorem A (ii): the fact that $\max_{x \in \mathbf{R}} u(x, t) \leq \max_{x \in \mathbf{R}} \bar{u}(x, t; M_0)$ for every $t > 0$ follows easily

from two properties: (i) u and \bar{u} have the same mass, M_0 ; (ii) $v_{xx} \geq -((m+1)t)^{-1}$ in $\Omega[u]$, $\bar{v}_{xx} = -((m+1)t)^{-1}$ in $\Omega[\bar{u}]$.

The rest is based on a precise description of the outer interfaces. We begin by revisiting the second-order differential inequality for the ζ_i 's obtained by Caffarelli and Friedman [11]:

LEMMA 3.1. *There exist nonnegative measures μ_i , $i = 1, 2$, on $(0, \infty)$ such that*

$$(3.1) \quad \zeta_i''(t) + (m/(m+1)t)\zeta_i'(t) = \mu_i(t)(-1)^i$$

in the sense of distributions. Hence the expression $(-1)^i \zeta_i'(t)t^{m/(m+1)}$ is nondecreasing in $(0, \infty)$.¹

REMARKS. (1) The coefficient $m/(m+1)$ is best possible: the self-similar solutions $\bar{u}(x, t; M, a)$ satisfy (3.1) with $\mu_i = 0$.

(2) Caffarelli and Friedman's result states (3.1) in the form $\zeta_i''(t) + k\zeta_i'(t) = \mu_i(t)(-1)^i$ with a constant $k > 0$. But the specification of k as $m/(m+1)t$ plays a fundamental role in the sequel for it gives the monotonicity of $\zeta'(t)t^{m/(m+1)}$.

PROOF. We review the proof in [11] to point out how k may be replaced by $m/(m+1)t$.

Let us take the case $i = 2$ and drop the i for simplicity. At a point of the interface $(\zeta(t_0), t_0)$ with $t_0 > t^*$, we adapt a self-similar solution $\bar{u} = \bar{u}(x - x_1, t; M_1)$ with x_1, M_1 so chosen as to have (i) $\zeta(t_0) = \bar{\zeta}(t_0)$ and (ii) $\zeta'(t_0) = \bar{\zeta}'(t_0)$, i.e. $v_x(\zeta(t_0), t_0) = \bar{v}_x(\zeta(t_0), t_0)$. Since $v_{xx} \geq -((m+1)t)^{-1} = \bar{v}_{xx}$ in $\Omega[\bar{u}]$, it follows for $t \geq t_0$ that $u(x, t) \geq \bar{u}(x, t)$. As in [11] we conclude that for $h > 0$,

$$(3.2) \quad \zeta(t_0 + h) - \zeta(t_0) - \zeta'(t_0)h \geq \bar{\zeta}(t_0 + h) - \bar{\zeta}(t_0) - \bar{\zeta}'(t_0)h.$$

Using the fact that for the self-similar solution \bar{u} ,

$$(3.3) \quad \bar{\zeta}''(t_0) + (m/(m+1)t_0)\bar{\zeta}'(t_0) = 0,$$

the second-member of (3.2) equals $-(h^2/2) \cdot (m/(m+1)t_0)\bar{\zeta}'(t_0) + O(h^3)$.

Now define the function Φ_h for $h > 0$ fixed by

$$(3.4) \quad \Phi_h(t_0) = \frac{\zeta(t_0 + h) - \zeta(t_0) - h\zeta'(t_0)}{h^2/2} \geq -\frac{m}{(m+1)t_0}\zeta'(t_0) + O(h).$$

One proves as in [11] that a subsequence of Φ_h converges weakly towards a signed measure so that in the limit (3.5) gives (dropping the zeros)

$$(3.5) \quad \zeta''(t) + (m/(m+1)t)\zeta'(t) = \mu$$

in the distribution sense in (t^*, ∞) and μ is nonnegative. Now divide (3.5) by $\zeta'(t) > 0$ to get

$$(3.6) \quad \lg(\zeta'(t)t^{m/(m+1)})' = \mu(t)/\zeta'(t) \geq 0.$$

Therefore $\zeta'(t)t^{m/(m+1)}$ is nondecreasing in (t^*, ∞) . But since $\zeta(t) = a_2$ in $0 \leq t \leq t^*$, the assertions of the lemma hold in $(0, \infty)$. #

¹ If $t = t_i^*$, $\zeta_i'(t_i^*)$ means the right derivative $\zeta_i'(t_i^* +)$.

We recall that for the self-similar solution $\bar{u} = \bar{u}(x, t; M_0)$, the interfaces are given by $(-1)^i \bar{\xi}_i(t) = r(t) \equiv c_m (M_0^{m-1} t)^{1/(m+1)}$. Combining Corollary 2.3 and Lemma 3.1 we obtain

LEMMA 3.2. For $t > t_i^*$ we have

$$(3.7) \quad (-1)^i \xi'_i(t) \leq r'(t) \quad \text{and} \quad (-1)^i \xi'_i(t)/r'(t) \uparrow 1 \quad \text{as } t \rightarrow \infty.$$

There exist b_i , $i = 1, 2$, such that $a_1 \leq b_1 \leq a_2$, and as $t \rightarrow \infty$,

$$(3.8) \quad \xi_2(t) - r(t) \downarrow b_2, \quad \xi_1(t) + r(t) \uparrow b_1.$$

Finally

$$(3.9) \quad \lim_{t \rightarrow \infty} t |\xi'_i(t) - (-1)^i r'(t)| = 0.$$

PROOF. We may consider only the case $M_0 = 1$ and $i = 2$. Also we drop the i 's.

Since $\xi'(t)t^{m/(m+1)}$ is nondecreasing (Lemma 3.1) there exists the limit $\lim_{t \rightarrow \infty} \xi'(t)t^{m/(m+1)} = K \leq \infty$. Since $\lim_{t \rightarrow \infty} \xi(t)t^{-1/(m+1)} = c_m$ (Corollary 2.3) we conclude that $K = c_m/(m+1)$ and (3.7) is proved.

In particular we have $\xi'(t) \leq r'(t)$ so that $\eta(t) = \xi(t) - r(t)$ is nonincreasing in t . Since $a_1 < \eta(t) < a_2$, (3.8) follows.

To prove (3.9) write (3.1) in the form

$$(3.10) \quad (t\xi')' = \xi'/(m+1) + t\mu(t).$$

Integrating in t gives

$$(3.11) \quad t\xi'(t) = \frac{\xi(t) - a}{m+1} + \int_0^t t d\mu(t).$$

Let $\xi(t) = \int_0^t t d\mu(t)$. (3.11) can be written as $t\eta'(t) = (\eta(t) - a_2)/(m+1) + \xi(t)$ so that (3.9) is equivalent to $\xi(\infty) = (a_2 - b_2)/(m+1)$. If this is not true and, say, $\xi(\infty) \geq (a_2 - b_2)/(m+1) + \varepsilon$ for an $\varepsilon > 0$, then we would have $\lim t\eta'(t) \geq \varepsilon$ as $t \rightarrow \infty$ so that $\lim \eta(t)/\lg t \geq \varepsilon$. But since $|\eta(t)| \leq \max(|a_1|, |a_2|)$, this is not possible. The same argument holds if $\xi(\infty) < (a_2 - b_2)/(m+1)$. #

REMARK. (3.11) implies that $(-1)^i(\xi_i(t) - a_i)t^{-1/(m+1)}$ is monotone nondecreasing. Hence we can formulate (2.9) more precisely:

$$(3.12) \quad (-1)^i(\xi_i(t) - a_i)t^{-1/(m+1)} \uparrow c_m M_0^{(m-1)/(m+1)}. \quad \#$$

To obtain the asymptotic expression (0.10) for $\xi_i(t)$ we need yet to show that $b_1 = b_2 = x_0$. We introduce the following expression: $d(t) = l(t) - 2r(t)$, where $l(t) = \xi_1(t) + \xi_2(t)$ is the dispersion of u at time t . (3.7) says that $d'(t) \leq 0$ and (3.8) that $d(t) \downarrow b_2 - b_1$ as $t \rightarrow \infty$. We show next that $b_1 = b_2$:

LEMMA 3.3. $b_1 = b_2$, i.e., there exists $b \in (a_1, a_2)$ such that as $t \rightarrow \infty$,

$$(3.13) \quad \xi_i(t) = (-1)^i r(t) + b + o(1).$$

PROOF. We divide the proof in two parts.

(I) We prove first that $b_2 \geq b_1$, i.e. that $\lim_{t \rightarrow \infty} d(t) \geq 0$. For that we evaluate ν at a fixed $t > 0$. Since $\nu(\xi_1(t), t) = 0$,

$$\nu_x(\xi_1(t), t) = -\xi'_1(t) = (r(t) - \varepsilon(t))/(m+1)t$$

where $\varepsilon(t) \geq 0$, $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$, and $\nu_{xx} \geq ((m+1)t)^{-1}$, we have

$$(3.14) \quad \nu(x, t) \geq -\frac{\hat{x}^2}{2(m+1)t} + \frac{(r(t) - \varepsilon(t))\hat{x}}{(m+1)t} \quad \text{if } \hat{x} \geq 0,$$

where $\hat{x} = x - \zeta_1(t)$. Hence $\nu(x, t) = 0$, $x > \zeta_1(t)$ implies $\hat{x} \geq 2(r(t) - \varepsilon(t))$. But since $\nu(x, t) = 0$ for $x = \zeta_2(t)$, i.e. for $\hat{x} = l(t) = 2r(t) + d(t)$, we conclude that $d(t) \geq -2\varepsilon(t)$. Let $t \rightarrow \infty$ to conclude.

(II) We now prove that $b_1 > b_2$ cannot occur.

The idea is to compare u at fixed times $t \gg 0$ with the self-similar solutions $\tilde{u} = \tilde{u}(x - x^*(t), t; M_0)$, $x^*(t) = \frac{1}{2}(\zeta_1(t) + \zeta_2(t))$ (i.e. the one centered in $\Omega(t)$) and estimate the integrals at time t :

$$(3.15) \quad \begin{aligned} I_1(t) &= \int_{[u < \tilde{u}]} (\tilde{u}(x, t) - u(x, t)) dx, \\ I_2(t) &= \int_{[u > \tilde{u}]} (u(x, t) - \tilde{u}(x, t)) dx. \end{aligned}$$

Since $\int u(x, t) dx = \int \tilde{u}(x, t) dx = M_0$, $I_1(t) = I_2(t)$ for every t . Nevertheless we shall show that if $b_2 > b_1$, I_2 is asymptotically larger than I_1 .

We begin by defining in $\{(x, t): t > T^*, \zeta_1(t) \leq x \leq \zeta_2(t)\}$ the function

$$(3.16) \quad f(x, t) = \nu_x(x, t) + (x - x^*(t))/(m+1)t$$

(cf. §1.4). We have

$$(3.17) \quad \partial f(x, t)/\partial x = \nu_{xx}(x, t) + ((m+1)t)^{-1} \geq 0$$

so that f is nondecreasing in x . Also

$$f(\zeta_1(t), t) = -\zeta_1'(t) - \frac{l(t)}{2(m+1)t} = \frac{d(t) - 2\varepsilon(t)}{2(m+1)t}.$$

Since $0 \leq d(t) \leq l_0$ we conclude that $f(\zeta_1(t), t) = O(1/t)$ as $t \rightarrow \infty$. Similarly $f(\zeta_2(t), t) = O(1/t)$. This and (3.17) give

$$(3.18) \quad f(x, t) = O(1/t) \quad \text{as } t \rightarrow \infty \text{ uniformly in } x \in \Omega(t).$$

Next we estimate $\nu(x, t) - \tilde{\nu}(x, t)$ for large t : for every $t \geq T^*$ and $x: |x - x^*(t)| \leq r(t)$ we have $\partial(\nu - \tilde{\nu})/\partial x = f$ so that

$$(3.19) \quad \begin{aligned} |\nu(x, t) - \tilde{\nu}(x, t)| &\leq |\nu(x^*(t) - r(t), t)| + \int_{x^*-r}^x |f(x, t)| dx \\ &\leq \sup |\nu_x| \cdot d(t)/2 + O(1/t) \cdot r(t) = O(r(t)/t) \end{aligned}$$

and the same estimate clearly holds for $\zeta_1(t) \leq x \leq \tilde{\zeta}_1(t)$ and $\tilde{\zeta}_2(t) \leq x \leq \zeta_2(t)$. (The estimate for $|\nu_x|$ comes from (3.16).) Hence we conclude that

$$(3.20) \quad |\nu(x, t) - \tilde{\nu}(x, t)| = O(t^{-m/(m+1)}) \quad \text{as } t \rightarrow \infty \text{ uniformly in } x \in \mathbf{R}.$$

From this it follows that uniformly in x such that $|x - x^*(t)| \leq \alpha r(t)$, $0 < \alpha < 1$, if $m > 2$ or in $x \in \mathbf{R}$ if $1 < m \leq 2$:

$$(3.21) \quad |u(x, t) - \tilde{u}(x, t)| = O(t^{-2/(m+1)})$$

(a simple application of the Mean Value Theorem since $u = ((m-1)\nu/m)^{1/(m-1)}$).

We are now in a position to estimate I_1 and I_2 under the hypothesis that $b_2 > b_1$: since $\int u(x, t) dx = \int \tilde{u}(x, t) dx$ and $l(t) - \tilde{l}(t) = 2d(t) > 0$, the set $G(t) = \{x \in \mathbf{R}: v(x, t) < \tilde{v}(x, t)\}$ is nonvoid. Let $x_1(t) = \inf G(t)$, $x_2(t) = \sup G(t)$. We have $\xi_1(t) \leq x_1(t) \leq x_2(t) \leq \xi_1(t)$. Since

$$(3.22) \quad \tilde{v}(x, t) = (r(t)^2 - |x - x^*(t)|^2)/2(m+1)t$$

for every $x: |x - x^*(t)| \leq r(t)$, recalling (3.14) we obtain, if $\lim d(t) = d > 0$, the estimate

$$(3.23) \quad x^*(t) - x_1(t) = o(r(t)),$$

and the same applies to $x_2(t)$. Thus as $t \rightarrow \infty$,

$$(3.24) \quad \begin{aligned} I_1(t) &= \int_{[u < \tilde{u}]} |\tilde{u} - u| dx \\ &\leq |x_2(t) - x_1(t)| \sup_{x_1(t) \leq x \leq x_2(t)} |u(x, t) - \tilde{u}(x, t)| \\ &\leq o(r(t)) \cdot O(r(t)^{-2}) = o(t^{-1/(m+1)}). \end{aligned}$$

In the following C will stand for any positive constant depending only on m . For large t it follows from (3.14), (3.22) that for $r(t)/3 < |x - x^*(t)| < r(t)/2$,

$$v(x, t) - \tilde{v}(x, t) \geq Ct^{-m/(m+1)},$$

so that we estimate I_+ from below:

$$(3.25) \quad I_+(t) \geq 2 \cdot (r(t)/6) \cdot Ct^{-2/(m+1)} \geq Ct^{-1/(m+1)}.$$

(3.24), (3.25) contradict the fact that $I_+(t) = I_-(t)$ for every t . Hence the assumption $b_2 > b_1$ was false. #

PROOF OF THEOREM A. Part (i) was proved in Corollaries 2.4 and 2.5. To prove part (ii) we repeat the calculations of the preceding lemma, taking into account the fact that $d(t) \downarrow 0$. Thus $x^*(t) = b + o(1)$ and (3.18), (3.20), (3.21) give, respectively:

$$(3.26) \quad v_x(x, t) + (x - b)/(m+1)t = o(1/t) \quad \text{uniformly in } x \in \Omega(t),$$

$$(3.27) \quad v(x, t) - \bar{v}(x, t; M_0, b) = o(t^{-m/(m+1)}) \quad \text{uniformly in } x \in \mathbf{R},$$

$$(3.28) \quad u(x, t) - \bar{u}(x, t; M_0, b) = o(t^{-2/(m+1)}),$$

uniformly in $x: |x - b| < \alpha r(t)$, $0 < \alpha < 1$, if $m > 2$ or in $x \in \mathbf{R}$ if $1 < m \leq 2$.

It only remains to prove that $b = x_0$:

If $1 < m \leq 2$ the proof is immediate from (3.28) and the invariance of the center of mass:

$$(3.29) \quad \begin{aligned} M_0 |x_0 - b| &= \left| \int x(u(x, t) - \bar{u}(x, t; M_0, b)) dx \right| \\ &= O(r(t)^2) \cdot o(t^{-2/(m+1)}) = o(1). \end{aligned}$$

Now let $t \rightarrow \infty$ to obtain $x_0 = b$.

Case $m > 2$. To simplify the calculation we may assume that $b = 0$ (if $b \neq 0$ shift the x -axis). Then

$$\begin{aligned}
 (3.30) \quad M_0 |x_0| &= \left| \int x(u - \bar{u}) dx \right| \\
 &\leq \int_{\xi_1(t)}^{-r(t)} |xu(x, t)| dt + \int_{r(t)}^{\xi_2(t)} |xu(x, t)| dt \\
 &\quad + \int_{-r(t)}^{r(t)} |x| |u(x, t) - \bar{u}(x, t)| dx \\
 &= I_1 + I_2 + I_3.
 \end{aligned}$$

Using (3.27) we estimate u in the region $[|x| \geq r(t)]$ as $o(t^{-m/(m^2-1)})$ so that

$$I_1 + I_2 = o(t^{-m/(m^2-1)}) \cdot r(t) \cdot d(t) = o(t^{-1/(m+1)}) \rightarrow 0.$$

Next we estimate I_3 . By (3.27) there is a function $\varepsilon(t) \geq 0$, $\varepsilon(t) \rightarrow 0$ such that for $|x| \leq r(t)$,

$$(3.31) \quad \bar{v}(x, t) - \varepsilon(t)t^{-m/(m+1)} \leq v \leq \bar{v} + \varepsilon(t)t^{-m/(m+1)}.$$

Let $F(t) = \{|x| \leq r(t): u(x, t) \geq \bar{u}(x, t)\}$. We have (C represents any constant > 0):

$$\begin{aligned}
 (3.32) \quad &\left| \int_{F(t)} x(u - \bar{u}) dx \right| \leq 2 \int_0^{r(t)} xC |v - \bar{v}|^{1/(m-1)} dx \\
 &\leq C \int_0^{r(t)} x dx \left\{ \left| \frac{r(t)^2 - x^2 + C\varepsilon(t)r(t)}{2(m+1)t} \right|^{1/(m-1)} - \left| \frac{r(t)^2 - x^2}{2(m+1)t} \right|^{1/(m-1)} \right\} dx \\
 &\leq Cr(t) \{ -C\varepsilon(t)r(t)^{-m/(m-1)} + (1 + C\varepsilon(t)/r(t))^{m/(m-1)} - 1 \} \\
 &= o(r(t)^{-1/(m-1)}) + o(1) \rightarrow 0.
 \end{aligned}$$

A similar computation applies on the set $G(t) = \{|x| \leq r(t): u(x, t) \leq \bar{u}(x, t)\}$ (by replacing $\varepsilon(t)$ by $-\varepsilon(t), \dots$) so that finally $I_3(t) \rightarrow 0$ and $x_0 = 0$ (i.e. $x_0 = b$) follows from (3.30) letting $t \rightarrow \infty$. #

4. Symmetric solutions. In this section we assume that $u_0(x)$ is symmetric, i.e.

$$(4.1) \quad u_0(x) = u_0(-x).$$

Since (0.1) is symmetry-invariant the corresponding solution $u(x, t)$ satisfies

$$(4.2) \quad u(x, t) = u(-x, t)$$

for every $x, t > 0$. We set $\xi(t) = \xi_2(t) = -\xi_1(t)$.

We introduce a comparison principle based on the estimate of the concentration of mass around the origin by means of which we prove optimal rates of convergence

in the results of Theorem A:

THEOREM B. *Let $u(x, t)$ be a solution of (0.1), (0.2) with u satisfying (0.3), (0.4), (4.1). Then there exist $\tau > 0$ and $t_1 \geq 0$ such that*

$$(4.3) = (0.10') \quad 0 \leq (\zeta(t) - r(t))t^{m/(m+1)} \leq k_1 \cdot M_0^{(m-1)/(m+1)} \tau \quad \text{if } t \geq t_1,$$

$$(4.4) = (0.11') \quad 0 \geq (\zeta'(t) - r'(t))t^{(2m+1)/(m+1)} \geq k_2 \cdot M_0^{(m-1)/(m+1)} \tau \quad \text{if } t \geq t_1,$$

$$(4.5) = (0.12') \quad \left| V(x, t) - \frac{x}{(m+1)t} \right| t^{(2m+1)/(m+1)} \leq k_2 M_0^{(m-1)/(m+1)} \tau$$

if $|x| \leq \zeta(t)$, $t \geq t_1$,

$$(4.6) = (0.13') \quad \left| v(x, t) - \frac{[r(t)^2 - x^2]}{2(m+1)t} \right| + t^{2m/(m+1)} \leq k_3 \cdot M_0^{2(m-1)/(m+1)} \tau$$

if $x \in \mathbf{R}$, $t \geq t_1$,

for some positive constants $k_1 = c_m/(m+1)$, $k_2 \leq k_1((m+1)/m)^m$, $k_3 \leq c_m(k_1/(m+1) + k_2)$. Moreover there exists $\tau_1 = \tau_1(m) > 0$ such that $\tau \leq \tau_1 M_0^{1-m} l_0^{m+1}$. #

REMARKS. (1) (4.6) and $u = ((m-1)v/m)^{1/(m-1)}$ give the estimate for u ,

$$(4.7) = (0.16') \quad u(x, t) = \bar{u}(x, t; M_0) + O(t^{-(m+2)/(m+1)}),$$

if $t \geq t_1$ uniformly in x : $|x| \leq \alpha r(t)$, $0 < \alpha < 1$ (if $1 < m \leq 2$ uniformly in $x \in \mathbf{R}$).

(2) The exponents in (0.10')–(0.14') are best possible. To see this apply the theorem to the explicit solution $u(x, t) = \bar{u}(x, t + \tau; M_0)$ for some $\tau > 0$, that serves as a model.

(3) We do not treat the problem of determining the best τ in Theorem B and its consequences. In this respect see the indications in [21].

(4) The proof of Theorem B can be adapted to treat radially-symmetric solutions in spatial dimension $n \geq 1$. #

The proof of Theorem B proceeds by comparing u with the self-similar solutions $\bar{u}(x, t; M_0)$ and $\bar{u}(x, t + \tau; M_0)$ for some suitable $\tau > 0$. As in §2 we first prove an elliptic version of the comparison principle.

LEMMA 4.1 (CONCENTRATION-COMPARISON PRINCIPLE. ELLIPTIC VERSION). *Let β be a continuous nondecreasing function such that $O = \beta(0) \subset \text{Int } \beta(\mathbf{R})$, let f_i , $i = 1, 2$, be symmetric and integrable functions in \mathbf{R} and define for $r \geq 0$,*

$$(4.8) \quad F_i(r) = \int_{|x| \leq r} f_i(x) dx.$$

Let u_i , $i = 1, 2$, be the (symmetric) solutions of $-u_i'' + \beta(u_i) = f_i$ [9], set $w_i = \beta(u_i)$ and for $r > 0$,

$$(4.9) \quad W_i(r) = \int_{|x| \leq r} w_i(x) dx.$$

Then if $F_1(r) \leq F_2(r)$ for every $r \geq 0$, then $W_1(r) \leq W_2(r)$ for every $r \geq 0$. #

REMARK. We say that f_2 is more concentrated than f_1 : $f_2 \succ f_1$. The lemma implies then that $w_2 > w_1$.

PROOF. As in Lemma 2.1 let $G = \{r \geq 0: W_1(r) > W_2(r)\}$. If G is nonvoid, let $I = (a, b)$ be a maximal interval in G , $0 \leq a < b \leq \infty$. As in Lemma 2.1 $u_1 - u_2$ is strictly increasing on I . Next if $b < \infty$ we have $W_1(b) = W_2(b)$, $u_1(b) \leq u_2(b)$ arguing as there and we conclude that $a = 0$ and $W_1(0) > W_2(0)$, impossible. The case $b = \infty$ is also similar. #

Via Semigroup Theory we pass to

LEMMA 4.2 (CONCENTRATION-COMPARISON PRINCIPLE. PARABOLIC VERSION). *Let $u^1(x, t)$, $u^2(x, t)$ be solutions of (0.1), (0.2) with symmetric initial data $u_0^1(x)$, $u_0^2(x) \in L^1(\mathbf{R})$. If for every $r > 0$,*

$$(4.10) \quad \int_{|x| \leq r} u_0^1(x) dx \leq \int_{|x| \leq r} u_0^2(x) dx,$$

then for every $t, r > 0$ we have

$$(4.11) \quad \int_{|x| \leq r} u^1(x, t) dx \leq \int_{|x| \leq r} u^2(x, t) dx. \quad \#$$

REMARK. Both lemmas admit obvious n -dimensional counterparts valid for radially-symmetric solutions. Hence Theorem B admits an n -dimensional version. #

COROLLARY 4.3. *Let $u(x, t)$ be a solution of (0.1)–(0.4), symmetric with respect to x and assume that there exist $\tau > 0$, $t_1 \geq 0$ such that $u(x, t_1) > \bar{u}(x, t_1 + \tau; M_0)$. Then for every $t \geq t_1$, $u(x, t) > \bar{u}(x, t + \tau; M_0)$ so that*

$$(4.12) \quad r(t) \leq \zeta(t) \leq r(t + \tau) \leq r(t) + r(t)\tau / ((m + 1)t).$$

PROOF. The inequality $r(t) \leq \zeta(t)$ comes from Theorem A. To obtain the inequality $\zeta(t) \leq r(t + \tau)$ we use Lemma 4.2 and the fact that $\zeta(t) = \sup\{r > 0: \int_{|x| \leq r} u(x, t) dx < M_0\}$. #

We now show that for every $u_0 \in L^1(\mathbf{R})$ satisfying (0.3), (0.4), (4.2) there exist $\tau > 0$ and $t_1 \geq 0$ such that (4.12) holds. Moreover we bound above τ in terms of M_0 and l_0 :

LEMMA 4.4. *There exists $\tau_1 > 0$ such that (4.12) holds with $\tau = \tau_1 \cdot M_0^{1-m} l_0^{m+1}$ for all large t , and $\tau_1 \geq (2c_m)^{-(m+1)}$.*

PROOF. By means of the group of transformations (1.9) we can reduce the proof to the case $M_0 = l_0 = 1$: if the lemma is true in this case and $u(x, t)$ is a general solution of (0.1)–(0.4), (4.2), we define \hat{u} by

$$(4.13) \quad u(x, t) = \frac{M_0}{l_0} \hat{u}\left(\frac{x}{l_0}, \frac{M_0^{m-1}}{l_0^{m+1}} t\right).$$

We have $\hat{M}_0 = \hat{l}_0 = 1$ so that $\hat{u}(\cdot, t) > \bar{u}(\cdot, t + \tau_1; 1)$ for all $t \geq \hat{t}_0 \geq 0$. Then

$$(4.14) \quad u(x, t) > \frac{M_0}{l_0} \bar{u}\left(\frac{x}{l_0}, \frac{M_0^{m-1}}{l_0^{m+1}} t + \tau_1; 1\right) = \bar{u}\left(x, t + \frac{\tau_1 l_0^{m+1}}{M_0^{m-1}}; M_0\right).$$

Therefore we assume that $M_0 = l_0 = 1$. Now note that there is a worst situation with respect to the relation " $>$ ", namely the one with initial condition

$$(4.15) \quad u_0(x) = \frac{1}{2}\delta(x - \frac{1}{2}) + \frac{1}{2}\delta(x + \frac{1}{2}).$$

We only have to prove that there exists a τ_1 for this particular u_0 (the fact that u_0 is a measure causes no inconvenience, see remark in §1.1). Since $u_0(x) \geq \frac{1}{2}\delta(x - \frac{1}{2})$, $u(x, t) \geq \bar{u}(x - \frac{1}{2}, t; \frac{1}{2})$ so that for every $t > t_0 = (4c_m^{m+1})^{-1}$ we have $u(x, t) \geq \bar{u}(\frac{1}{2}, t; \frac{1}{2}) > 0$ for $0 \leq x \leq 1$. Also the free-boundary ζ_2 of \bar{u} passes through $(1, t_0)$. By the Sh.C.P. we derive the estimate $1 < \zeta(t) \leq 1 + c_m(t - t_0)^{1/(m+1)}$. To obtain $u(x, t) > \bar{u}(x, t + \tau; 1)$ at a time $t > t_0$ we only have to take τ large enough, for instance such that

$$(4.16) \quad \bar{u}(0, t + \tau; 1) \cdot \zeta(t) \leq \bar{u}(\frac{1}{2}, t; \frac{1}{2}).$$

This $\tau = \tau(t, m)$ is to be minimized in $t > t_0$ to obtain τ_1 .

Consider now the explicit solution $u(x, t) = \bar{u}(x, t + \tau; 1)$ such that $l_0 = 1$; then $\tau = (2c_m)^{-(m+1)}$, so that $\tau_1 > (2c_m)^{-(m+1)}$. #

PROOF OF THEOREM B. Corollary 4.3 and Lemma 4.4 imply (4.3).

To prove (4.4) we restrict ourselves as above to the case $M_0 = l_0 = 1$. Let us estimate the derivative of $\eta(t) = \zeta(t) - r(t)$ using the fact that $t^{m/(m+1)}\eta'(t) \uparrow 0$ (Lemma 3.2). For $t > 0$, $\lambda > 1$ we have

$$\begin{aligned} \eta(t) &= \int_t^\infty -\eta'(s) ds \geq \int_t^{\lambda t} -\eta'(s) ds \geq -\eta'(\lambda t)(\lambda t)^{m(m+1)} \int_t^{\lambda t} s^{-m/(m+1)} ds \\ &= -\eta'(\lambda t)(m+1)\lambda^{m/(m+1)}t(\lambda^{1/(m+1)} - 1). \end{aligned}$$

Using (4.3) we obtain

$$(4.17) \quad -\eta'(t) \leq k_1 \tau \lambda \{ (m+1)t^{2(m+1)/(m+1)}(\lambda^{1/(m+1)} - 1) \}^{-1}.$$

The right-hand expression is minimized setting $\lambda = ((m+1)/m)^{m+1}$; then (4.18) gives (4.4) with $k_1 \leq k_2((m+1)/m)^m$.

Estimates (4.5), (4.6) are obtained by inserting the information (4.3), (4.4) into formulas (3.16)–(3.18) in Lemma 3.3. #

5. Other results on the growth of the interfaces. In this section we consider solutions $u(x, t)$ of the Cauchy problem (0.1), (0.2), where u_0 satisfies (0.3), and instead of (0.4) the half-condition

$$(0.18) \quad \text{ess sup}(\text{support}(u_0)) = 0.$$

Then a right free-boundary $x = \zeta(t)$, appears where $\zeta(t) = \{\sup x: u(x, t) > 0\}$ for $t > 0$, $\zeta(0) = 0$. By the Sh.C.P. $\zeta(t)$ is finite and in fact

$$(5.1) \quad 0 \leq \zeta(t) \leq c_m \|u_0\|_1^{(m-1)/(m+1)} t^{1/(m+1)}.$$

What was said in the Introduction applies and thus there exists a time $t^* \geq 0$ such that $\zeta(t) = 0$ for $0 \leq t \leq t^*$ and $\zeta \in C^1((t^*, \infty))$ and $\zeta'(t) > 0$ if $t > t^*$. Also $\zeta(t)t^{-1/(m+1)}$ and $\zeta'(t)t^{m/(m+1)}$ are nondecreasing in $(0, \infty)$. Furthermore we prove

THEOREM C. As $t \rightarrow \infty$ we have (with $r(t) = c_m(M_0^{m-1}t)^{1/(m+1)}$)

$$(5.2) \quad \zeta(t)/r(t) \uparrow 1, \quad \zeta'(t)/r'(t) \uparrow 1.$$

As $t \rightarrow 0$ we have

$$(5.3) \quad \zeta(t)/r(t) \downarrow 0, \quad \zeta'(t)/r'(t) \downarrow 0.$$

Moreover if $x_0 = M_0^{-1} \cdot \int u_0(x)x \, dx$, $-\infty \leq x_0 < 0$,

$$(5.4) \quad \zeta(t) - r(t) \downarrow x_0.$$

And if x_0 is finite, then

$$(5.5) \quad t(\zeta'(t) - r'(t)) \uparrow 0.$$

PROOF. Take the sequence of approximations to $u(x, t)$, $\{u^n(x, t)\}$ such that $u^n(x, t)$ is the solution of (0.1) with initial condition $u_0^n = u_0 \cdot \chi_{[-n, 0]}$. If $\zeta^n(t)$ is the corresponding right-interface it follows from $u(x, t) \geq u^n(x, t)$ that $\zeta(t) \geq \zeta^n(t)$. But by (2.9) $\zeta^n(t)t^{-1/(m+1)} \rightarrow c_m(M_n)^{(m-1)/(m+1)}$, where $M_n = \int_{-n}^0 u_0(x) \, dx$. Since $M_n \rightarrow M_0$ as $n \rightarrow \infty$, this and (5.1) give (5.2). To obtain $\zeta'(t)/r'(t) \rightarrow 1$ argue as in Lemma 3.2.

To prove (5.3) compare u with the solution u_ε resulting from shifting the initial mass lying in $[-\varepsilon, 0]$, for small $\varepsilon > 0$, to 0 as a point mass $M_\varepsilon \delta(x)$. In some time interval $[0, t_\varepsilon]$, $t_\varepsilon > 0$, the right-interface for u_ε coincides with the one for this point mass and the Sh.C.P. gives us

$$(5.6) \quad \zeta(t) \leq c_m(M_\varepsilon^{m-1}t)^{1/(m+1)} \quad \text{for } 0 \leq t < t_\varepsilon.$$

Now let $\varepsilon \rightarrow 0$: then $M_\varepsilon \rightarrow 0$ and (5.4) implies that $\zeta(t)/r(t) \rightarrow 0$. Since $\zeta'(t)t^{m/(m+1)}$ is monotone the limit $\zeta'(t)/r'(t)$ exists and is zero.

To prove (5.4) notice that there exists b , $-\infty \leq b \leq 0$, such that $\zeta(t) - r(t) \downarrow b$ since $\zeta'(t) \leq r'(t)$ for every $t > 0$. We shall prove that $b = x_0$. For that we call $\hat{u}^n(x, t)$ the solution resulting from shifting the mass of u_0 in $(-\infty, n]$ as a point mass to $x = -n$, keeping $\hat{u}_0^n(x) = u_0(x)$ for $x > -n$. Let $\hat{\zeta}^n(t)$ and \hat{x}_0^n be the corresponding right-interface and center of mass. The sequence $\{\hat{\zeta}^n(t)\}_{n \in \mathbb{N}}$ is nonincreasing in n by the Sh.C.P. and Theorem A says that $\hat{\zeta}^n(t) = \hat{x}_0^n + r(t) + o(1)$. Since $\hat{x}_0^n \downarrow x_0$ as $n \rightarrow \infty$, $b = \lim_{t \rightarrow \infty} (\zeta(t) - r(t)) \leq x_0$, so that in case $x_0 = -\infty$ we are done.

It remains to prove that $b \geq x_0$ in case $x_0 > -\infty$. Take an $\varepsilon > 0$. It is clear that there exist n_ε and t_ε such that for $n \geq n_\varepsilon$ and $t \geq t_\varepsilon$, $\hat{\zeta}^n(t) < x_0 + r(t) + \varepsilon$. Since $\hat{\zeta}^n(t)/r'(t)$ is nondecreasing (we drop the index n in this calculation),

$$\begin{aligned} \varepsilon &> \hat{\zeta}(t) - r(t) - \hat{x}_0 = \int_t^\infty (r'(s) - \hat{\zeta}'(s)) \, ds = \int_t^{2t} r'(s) \left[1 - \frac{\hat{\zeta}'(s)}{r'(s)} \right] ds \\ &\geq (r(2t) - r(t)) \cdot \left(1 - \frac{\hat{\zeta}'(2t)}{r'(2t)} \right) = r(2t)(1 - 2^{-1/(m+1)}) \left(1 - \frac{\hat{\zeta}'(2t)}{r'(2t)} \right). \end{aligned}$$

So that for $t \geq 2t_\varepsilon$, $1 - \hat{\zeta}'(t)/r'(t) \leq k_m \varepsilon / r(t)$ with $k_m > 0$ depending only on m . Since $-\hat{p}_x(\hat{\zeta}(t), t) = \hat{\zeta}'(t)$ [16], and $\hat{p}_{xx} \geq -((m+1)t)^{-1}$, we have at $\bar{x}(t) = x_0 + r(t) - \varepsilon$, for t large enough ($t \geq 2t_\varepsilon$, $2k_m \varepsilon \leq r(t)$):

$$\begin{aligned} \hat{p}(\bar{x}(t), t) &\geq \left(r'(t) - \frac{k_m \varepsilon r'(t)}{r(t)} \right) (\hat{\zeta}(t) - \bar{x}(t)) - \frac{(\hat{\zeta}(t) - \bar{x}(t))^2}{2(m+1)t} \\ &\geq \frac{1}{2} r'(t) \cdot \varepsilon - \frac{4\varepsilon^2}{2(m+1)t} \end{aligned}$$

so that $\hat{p}''(\bar{x}(t), t) \geq \varepsilon r'(t)/2$ for t large enough uniformly in n . Thus in the limit $v(\bar{x}(t), t) > 0$, so that $\zeta(t) \geq \bar{x}(t)$ for every large t , $\lim_{t \rightarrow \infty} (\zeta(t) - r(t) - x_0) > -\varepsilon$ and the result follows.

(5.5) follows from (5.4) arguing as in Lemma 3.2. #

(5.2) makes clear that for an u_0 satisfying (0.3), (0.4'), M_0 and x_0 allow us to describe $\zeta(t)$ as $t \rightarrow \infty$ in the first approximation. As $t \rightarrow 0$ (5.3) shows that this is not the case: The description of $\zeta(t)$ requires further information: thus Knerr [16] proves that if $u_0 \in L^\infty(\mathbf{R})$ and $\|v_0\|_\infty \leq L$ ($v_0 = mu_0^{m-1}/(m-1)$), $\zeta(t) \leq 2(Lt)^{1/2}$ for every small t and the exponent $\frac{1}{2}$ is sharp for this class of initial data.

We extend the result to cover the dependence of $\zeta(t)$ on the L^p -norm of u_0 for every $1 \leq p \leq \infty$: we consider the class of solutions

$$(5.7) \quad \mathcal{C}_{p,N} = \{u(x, t): u \text{ is solution of (0.1), (0.2), (0.3), (0.18)}\}$$

$$\text{with } u_0 \in L^p(\mathbf{R}) \text{ and } \|u_0\|_p \leq N\}$$

where $1 \leq p \leq \infty$, $N > 0$ and $\|\cdot\|_p$ denotes the L^p -norm. If $\zeta_u(t)$ is the right-interface of $u \in \mathcal{C}_{p,N}$, we set for $t \geq 0$,

$$(5.8) \quad \mathcal{L}_{p,N}(t) = \sup\{\zeta_u(t): u \in \mathcal{C}_{p,N}\}.$$

We have

THEOREM D. *For every p : $1 \leq p \leq \infty$ there exists a constant $C_{p,m} > 0$ such that*

$$(5.9) \quad \mathcal{L}_{p,N}(t) = C_{p,m} (N^{m-1}t)^\alpha$$

with $\alpha = p/(2p + m - 1)$ if $1 \leq p < \infty$, $\alpha = \frac{1}{2}$ if $p = \infty$. We have $C_{1,m} = c_m$ (c_m defined in (0.9)).

PROOF. By means of the group of transformations $T_{k,L}$ (§1.2) we can reduce the proof to the case $N = t = 1$. In fact let u be a solution with L^p -norm $N > 0$ and fix a certain $\bar{t} > 0$. If we define \hat{u} by

$$(5.10) \quad \hat{u}(x, t) = (T_{k,L}u)(x, t) = Ku(Lx, K^{m-1}L^2t)$$

with $L = (N^{m-1}\bar{t})^\alpha$ and $K = (N^{-2p}\bar{t})^\beta$ with $\beta = \alpha/p$ if $p < \infty$, $\beta = 0$ if $p = \infty$, then $\hat{u} \in \mathcal{C}_{p,1}$ and

$$(5.11) \quad \zeta(\bar{t}) = L\hat{\zeta}((k^{m-1}L)^{-1}\bar{t}) = (N^{m-1}\bar{t})^\alpha \cdot \hat{\zeta}(1).$$

Hence we must prove that $C_{p,m} = \sup\{\zeta_u(1): u \in \mathcal{C}_{p,1}\}$ is finite for every $m > 1$, $1 \leq p \leq \infty$.

For $p = 1$ it follows from Corollary 2.3 that $C_{1,m} = c_m$. On the other hand for $p = \infty$ [16] proves that $C_{\infty,m} \leq 2(m/(m-1))^{1/2}$. For $1 < p < \infty$ we estimate $C_{p,m}$ in terms of $C_{\infty,m}$ by means of the following “ L^p - L^∞ smoothing effect”:

LEMMA 5.1 [20]. *There exists a positive constant $K = K(m, p)$ such that for every solution of (0.1), (0.2) with $u_0 \in L^1(\mathbf{R}) \cap L^p(\mathbf{R})$, $1 \leq p < \infty$,*

$$(5.12) \quad \|u(t)\|_\infty \leq K \|u_0\|_p^\sigma t^{-\delta}$$

with $\sigma = 2p/(2p + m - 1)$, $\delta = 1/(2p + m - 1)$. #

Hence for every $t, h \geq 0$ and every $u \in \mathcal{C}_{p,1}$ we have

$$(5.13) \quad \zeta(t+h) \leq \zeta(t) + C_{\infty,m} (\|u(t)\|_\infty^{m-1} t)^{1/2}.$$

Fix $t > 0$ and set $t_n = 2^{-n}t$, $n \geq 1$. (5.12) and (5.13) give

$$(5.14) \quad \zeta(t_{n-1}) \leq \zeta(t_n) + C_{\infty,m} K^{(m-1)/2} (2^{-n}t)^{p/(2p+m-1)},$$

and since $\lim_{n \rightarrow \infty} \zeta(t_n) = \zeta(0) = 0$, we conclude from (5.14) that

$$(5.15) \quad \zeta(t) \leq C_{\infty,m} (K^{(m-1)/2} / (2^\alpha - 1)) t^\alpha$$

with $\alpha = p(2p + m - 1)^{-1}$. Therefore

$$(5.16) \quad C_{p,m} \leq C_{\infty,m} \cdot K^{(m-1)/2} \cdot (2^\alpha - 1)^{-1}. \quad \#$$

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